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Plug-flow/dispersion model of longitudinal dispersion

Zhou Kezhao

Southwest Municipal Engineering Design Institute
of China, Chengdu 610081, China

Abstract — A modified Fickian plug-flow/dispersion model (P/D model) is developed in this study. In P/D model, the flow process is divided into two belts, plug flow belt and dispersion belt. P/D model is very similar to Fickian model and rather perfect. The prediction by P/D model can be always consistent with experimental data in river, flume, and pond, even though the data are much skew. Therefore, P/D model is better than Fickian model and other dispersion models.

Keywords: plug-flow/dispersion model; dispersion model; Fickian model; longitudinal dispersion; water quality.

INTRODUCTION

Dispersion is the mixing or spreading caused by advective velocity variation within cross section, the dispersion process is often described by one-dimensional Fickian diffusion equation, i. e.

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2} , \tag{1}$$

where C=cross-sectional mean concentration, u=advective velocity, t=time, x=longitudinal distance, D=longitudinal dispersion coefficient.

Fickian dispersion model is not perfect. With the deepening of the research, its defects are discovered. Nordin and Sabol (1974) reported that there are more deviation between the prediction by Fickian model and the practical dispersion process in river. Day (1975) and other investigators (Fischer, 1979) also pointed out this problem. Therefore, many investigators (McQuivey, 1976; Valentine, 1977; Harden, 1979; Stefan, 1981; Beer, 1983) have attempted to improve upon Fickian model or to develop other dispersion models. Nevertheless, because of too many parameters and difficult calculations of these non-Fickian models, they are not perfect yet and not used widely.

Therefore, this paper attempts to reserve the advantages of only one parameter and simple equation of Fickian model, and improves upon it to describe the skew time-concentration distribution and let it consist with test data perfectly. Through theoretical analysis and computer simulation to numerous test data in river, flume, and pond, the improvement is achieved successfully by dividing the whole river reach into plug flow belt and dispersion belt.

In Fickian model Equation (1), the advective velocity u and longitudinal dispersion coefficient D must be constants in whole dispersion process, but this requirement can not be met usually, so it can not applied in uneven waters and in initial phase or initial time, so it is inconvenient in use. In natural waters (river, lake, and so on) and artificial structures (pond, tank and so on), thc flow is not often uniform. Eventhough the flow is uniform, the tracer cloud is spreaded within the cross section gradually after tracer released, i. e., the instant face source can not be made in practice, so the velocity of tracer cloud and the dispersion capacity of the waters are also variable in dispersion process. Because of these, Fickian model can be used only in uniform flow, and only after initial phase or initial time. Its deviation can not be accepted in many conditions too. In order to develop improved model to fit practical dispersion process, the supposition of constant u and D must be given up.

THEORETICAL FRAMEWORKS

According to the universal three-dimensional turbulent diffusion equation:

$$\frac{\partial c}{\partial t} - u_x \frac{\partial c}{\partial x} + u_y \frac{\partial c}{\partial y} + U_z \frac{\partial C}{\partial z} = \frac{\partial}{\partial x} (D_{tx} \frac{\partial c}{\partial x}) + \frac{\partial}{\partial y} (D_{ty} \frac{\partial c}{\partial y}) + \frac{\partial}{\partial z} (D_{tz} \frac{\partial c}{\partial z}) + S, \quad (2)$$

where c=concentration; t=time; x, y, z=longitudinal, lateral and vertical coordinates individually; u_x, u_y, u_z=longitudinal, lateral and vertical velocities separately; D_{tx}, D_{ty}, D_{tz}=longitudinal, lateral, and vertical turbulent diffusion coefficients separately; S=source or sink.

If the contaminant is stable, considering the flow uniform in a very short segment approximately, computing the cross-sectional mean values of all terms in Equation (2), applying integral intermediate value theorem, and noting no turbulent transport at boundary of waters, then, Equation (2) can be simplified to

$$\frac{\partial C}{\partial t} + u_x(\xi_1) \frac{\partial C}{\partial x} = D_{tx}(\xi_2) \frac{\partial^2 C}{\partial x^2} , \quad (3)$$

where C=cross-sectional mean concentration, $u_x(\xi_1)$=longitudinal velocity at a certain point ξ_1 within cross section as a characteristic advective velocity in this very short segment, $D_{tx}(\xi_2)$=longitudinal turbulent diffusion coefficient at a certain point ξ_2 within cross section as a characteristic longitudinal dispersion coefficient in this short enough segment.

Equation (3) is an important result, which indicates that if the flow is uniform in a segment only, the Fickian dispersion equation can be established certainly, in spite of the fact that the velocity in the equation may be not the real advective velocity.

As shown in Fig. 1, the reach from release section to sampling section is divided into n segments. In these very short segments, the flow can be considered uniform approximately.

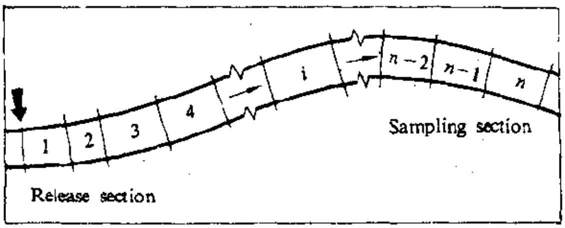

Fig. 1 Segments of a reach

In a short enough segment i, writing $u_x(\xi_1)$ and $D_{tx}(\xi_2)$ as u_i and D_i separately. Let

$$u_i=\alpha_i u, \quad D_i=\beta_i D. \tag{4}$$

And let

$$dT=\frac{\alpha_i^2}{\beta_i}\,dt, \quad dX=\frac{\alpha_i}{\beta_i}\,dx. \tag{5}$$

where u=mean advective velocity in whole reach; D=mean longitudinal dispersion coefficient in whole dispersion process; α_i and β_i=dimensionless coefficients, the subscript i shows segment i.

Therefore, Equation (3) can be simplified to

$$\frac{\partial C}{\partial T}+u\frac{\partial C}{\partial X}=D\frac{\partial^2 C}{\partial X^2}\,. \tag{6}$$

This is a very interesting result.

Equation (6) is the same as Fickian model Equation (1). In Equation (6) parameters u and D are independent of i, so u and D are constants throughout whole reach from release section to sampling section. It seems that Fickian model can be applied in whole reach. But, it is well known that Fickian model can not be applied in whole reach, if the flow is uniform only in a series of very short segments respectively.

It must be noted that the scales of time and distance in Equation (6) may be changed. For example, the time scale dt and distance scale dx in segment i have been replaced by d_T and dX, or $(\alpha_i^2/\beta_i)dt$ and $(\alpha_i/\beta_i)dx$.

From Equation (4) and (5), it can be found that

$$d_T=\frac{u_i^2/u^2}{D_i/D}\,dt, \qquad dX=\frac{u_i/u}{D_i/D}\,dX\,. \tag{7}$$

As judged by Equation (7), the time scale d_T and the distance scale dX in segment i are lengthened with u_i increasing or D_i decreasing, so the length of time and distance are shortened, and conversely, the length of time and distance are lengthened with u_i decreasing or D_i increasing, only if $u_i = u$ and $D_i = D$ the lengthes of time and distance are not changed.

As the length of time and distance in any segment may be changed, so the length of time and distance of whole reach may be changed, too. The differences of changed length of time and distance from the reality those can be explained to the release time and position of imaginal source separately (Zhou, 1987), or the travel time and distance in plug flow belt separately in this reach. Because the physical meaning of the latter is more intuitional, so author select the latter now rather than the before.

This is the plug flow/dispersion model (P/D model), of which the equation is the same as Fickian model, but the whole reach is divided into plug flow belt and dispersion belt. The longitudinal dispersion coefficient is *0* in plug flow belt, or D in dispersion belt. The advective velocity is u in two belts. The advective time and distance are experssed in τ and χ individually in plug flow belt, or $(t-\tau)$ and $(x-\chi)$ individually in dispersion belt.

According to the practice of the river, both belts can be divided into many segments respectively, the segments of any belt can be arranged continuously or alternately with those of other belt. The pattern of arrangement can not affect the dispersion result. For example, the shoals and bays in Tuo River can be considered as the segments of plug flow belt and dispersion belt individually. Tuo River is a winding river with many shoals and bays alternately. The flow in shoals is very narrow, shallow, and rapid. The flow in bays is very wide, deep, and slow.

In above theory development, the source intensity is not involved, of course, the source intensity of P/D model need not be changed.

Obviously, χ is relative with τ, i. e.

$$\chi = u\tau \ , \tag{8}$$

therefore, P/D model only has one parameter, χ or τ, more than Fickian model.

In the theoretical analysis about P/D model, the supposition that flow is uniform in any short enough segment is applied, which fits the conditions of various waters such as river, flume, and pond, more easily than the supposition required by Fickian model. Therefore, P/D model can be applied to uneven waters, but Fickian model can not. Moreover, the supposition causing the limitation of initial phase or initial time of Fickian model is not applied to P/D model. Of course, the limitation of initial phase or initial time no longer exists in P/D model. P/D model is very simple, which has analytic solution corresponding with instantaneous source, whether the river is uniform. Therefore, P/D model is better than Fickian model for dispersion process in various waters.

COMPARISON BETWEEN *P/D* MODEL AND FICKIAN MODEL

The equations, parameters and some important results of two models corresponding with instantaneous plane source are listed in Table 1.

Table 1 Comparison between P/D model and Fickian model corresponding with instantaneous plane source

	P/D model	Fickian model
Equation	$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}=D\frac{\partial^2 C}{\partial x^2}$	
Source intensity	M	
Release time	0	
Release distance	0	
Distance length in reach	x	x
Plug flow belt	χ or $u\tau$	
Dispersion belt	$x-\chi$ or $x-u\tau$	
Travel time in reach	x/u	x/u
Plug flow belt	τ or χ/u	
Dispersion belt	$x/u-\tau$ or $(x-\chi)/u$	
Advective velocity	u	
Longitudinal dispersion coefficient in reach		D
Plug flow belt	0	
Dispersion belt	D	
Concentration	$\frac{M}{\sqrt{4\pi D(t-\tau)}}\exp\left[-\frac{(x-ut)^2}{4D(t-\tau)}\right]$	$\frac{M}{\sqrt{4\pi Dt}}\exp\left[-\frac{(x-ut)^2}{4Dt}\right]$
Centre time	$2D/u^2+x/u$	
Time variance	$8D^2/u^4+2D(x-\chi)/u^3$	$8D^2/u^4+2Dx/u^3$
Skew coefficient	$\frac{64+12u(x-\chi)/D}{[8+2u(x-\chi)/D]^{3/2}}$	$\frac{64+12ux/D}{[8+2ux/D]^{3/2}}$

By the way, the results about Fickian model in Table 1 have been proposed (Zhou, 1986; 1987; 1988), which can be used to solve many outstanding questions in Fickian model application and develop some new improved methods to evaluate longitudinal dispersion coefficient.

From Table 1, it can be found that all skew coefficients of two models take the same maximum $2\sqrt{2}$ where $x=\chi$, and $x=0$ separately, both decrease gradually with the increasing of x, and both take the same minimum 0 where $x \to \infty$. If the longitudinal dispersion coefficients of two models are the same, the skew coefficient of P/D model would be bigger than that of Fickian model in same x (Fig. 2). For this reason, P/D model can fit much skew time-concentration distribution data taken by tracer tests in various waters more easily than Fickian model.

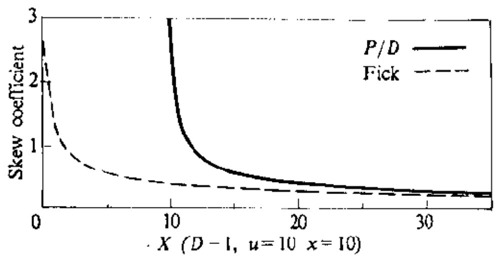

Fig. 2 Comparison between skew coefficients of *P/D* model and Fickian model

It should be indicated that, if $\chi=0$ or $\tau=0$, P/D model would degenerate into Fickian model. Consequently, P/D model is the extension of Fickian model, and Fickian model is a special case of P/D model.

PARAMETER ESTIMATION

The parameters of P/D model, D, χ or τ, u, and M can be evaluated directly by Gauss-Newton's method or McQuardt's method according to tracer test data.

This nonlinear approximation is a more accurate, direct, rapid, and well-fitting method than others. Perhaps, it is the best method of longitudinal dispersion coefficient evaluation (Zhou, 1986).

VERIFICATION OF P/D MODEL

P/D model has been verified with tracer test data taken in river, flume, and pond.

River field tracer tests were held at Tuo River, Sichuan, March to April, 1984. Experimental reach is over 200 km long. In these tests, release Rodanmine *B* tracer, 7 times, a total over 100 kg. There are 23 sampling sections, and many different position and depth sampling points in each sampling section.

Flume tracer tests were undertaken at Hydraulic Laboratory, Tsinghua University, from October 1984 to April 1985. Test flume is 21.6 m long and 0.6 m wide. The tracer is salt. These 111 times of tracer tests consist of smooth bed tests, rough bed tests, and two types of shoal/bay bed tests.

Pond tracer tests were undertaken at Southwest Municipal Engineering Design Institute of China, Chengdu, April 1988 to January 1989. Test pond is an oxidation pond used for wastewater stabilization lagoon research. Tracer is calcium chloride. These 82 times of tracer tests consist of 17 types of influent and effluent conditions, and many different discharge, shape, L/W, and depth conditions in each type.

All 231 sets of time-concentration data are approximated by nonlinear approximation according to P/D model and Fidkian model. Three examples of river, flume, and pond are shown in Fig. 3 to Fig. 5.

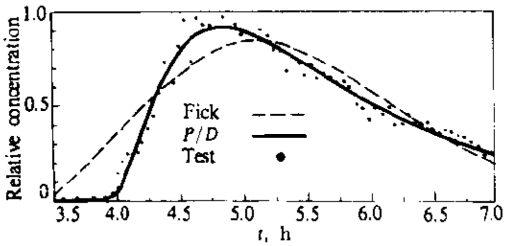

Fig. 3 Results predicted by P/D model and Fickian model with river tracer test data

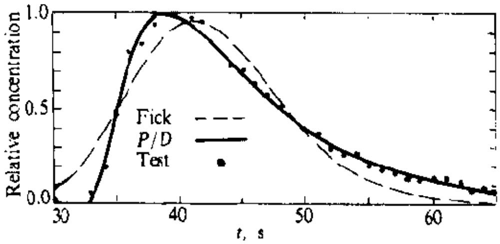

Fig. 4 Results predicted by P/D model and Fickian model with flume tracer test data

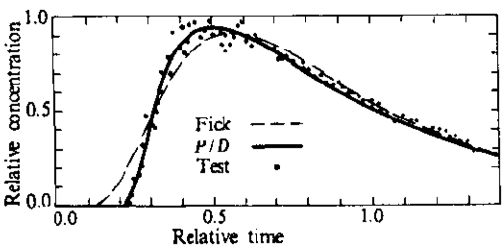

Fig. 5 Results predicted by P/D model and Fickian model with pond tracer test data

As shown in Fig. 3 to Fig. 5, the curves predicted by P/D model can corresponded with the tracer test data much more excellently than those by Fickian model. It is indicated by approximation results that P/D model can be successfully used in both natural waters as river or flume and artificial structures as pond.

CONCLUSION

The P/D model proposed in this paper is a new improvement on Fickian model. It maintains the simple equation of Fickian model, but divides the dispersion process into plug flow belt and dispersion belt. It has analytic solution corresponding with instantaneous source, in spite of whether or no the waters as river, flume, and pond are uniform. It can fit tracer test data much more excellently than Fickian model, even they are very skew. It is very simple, has only one parameter more than Fickian model. Therefore, P/D model is better than Fickian model and other dispersion models.

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