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# A numerical method for spherical particle motion in turbulent flow with large Reynolds-number

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**Abstract**: A new mathematical model, fluctuation spectrum random trajectory model (FSRTM) for the particle motion in environmental fluid was developed using Lagrangian method, in which the time-mean velocity of the fluid was calculated by a time-mean velocity formula for two dimensional homogeneous shear turbulent flows in open channel, the velocity fluctuation of the fluid was determined by Fourier expansion and fluctuation spectrum, and the particle motion equation was solved using Ronge-Kutta method. For comparison, the spherical cation exchange resins with a density of 1.44 g/cm$^3$ and diameters ranging from 0.50—0.60 mm, 0.60—0.70 mm and 0.80—0.90 mm were selected as the experimental solid particles, and their moving velocities and trajectories in shear turbulent flows with the flow Reynolds number of 4710, 10240, 11900 and 20760 were investigated. The comparing analyses of the modeled results with the measured results have shown that the model developed in this paper can describe the motions of the particles in shear turbulent flow.

**Key words**: shear turbulent flow; particle motion; fluctuation spectrum random trajectory model

**CLC number**: X11 **Document code**: A

## Introduction

The solid/fluid two-phase flows are often encountered in many environmental, chemical and hydraulic engineering, such as sediment transportation in rivers, waste water flow with solid particles and so on. Recently, considerable efforts regarding numerical predictions have been made, where the methods may be divided into two categories, namely the Eulerian approach and the Lagrangian approach. For the motion of the dispersed particle in turbulent flows, BBO(Basset-Boussinesq-Oseen) equation has been used widely(Hjelmfet, 1969; Liang, 1984; Shen, 1989). However, many studies have shown that BBO equation can only be used in the flows with lower relative Reynolds numbers of particle and fluid, and in which the drag acting on the particle must obey the Stokes drag law(Odar, 1964). The flows in nature often have the large relative Reynolds number, BBO equation can not describe the real motion of suspended particle in the flows. In this paper, a particle motion equation is developed, in which the effect of the relative Reynolds number on the drag, and the dependent relationships of added mass force and Basset history force on the acceleration modulus of particle are taken into account(Zhang, 1997).

In the study of the mathematical models describing particle motion in turbulent flows, Jurewicz(Jurewicz, 1976) first proposed the modifying method to the particle drifting velocity and drifting force; based on it, Smith (Smith, 1981) divided arbitrarily the particle mean velocity into convection velocity and drifting velocity. Because it is difficult to determine the particle diffusivity due to the shortage of the experimental data, the wide applications of this method are restricted. In addition, the semi-random trajectory model of particle is used widely(Yoshida, 1980; Fan, 1984; Li, 1994), in which the fluctuation velocity of fluid is calculated based on the root-mean-square(r.m.s) of fluctuation velocity. In fact, the turbulence is composed randomly of the eddies with different frequencies, amplitudes and phase angles, so the r.m.s can not reflect fully the essence of turbulent eddies. Cen(Cen, 1990) developed a fluctuation spectrum random trajectory model to describe the motions of particle in gas. In this paper, the fluctuation spectrum random trajectory

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model (FSRTM) for the particle motion in environmental fluid is developed using Lagrangian method, in which the time-mean velocity of the fluid is calculated by a time-mean velocity formula for two dimensional homogeneous shear turbulent flows in open channel, the velocity fluctuation of fluid is determined by Fourier expansion and fluctuation spectrum, and the particle motion equation is solved using Ronge-Kutta method.

For comparison, the spherical cation exchange resins with a density of 1.44 $g/cm^3$ and diameters ranging from 0.50—0.60 mm, 0.60—0.70 mm and 0.80—0.90 mm are selected as the experimental solid particles, and their moving velocities and trajectories in shear turbulent flows with the flow Reynolds number of 4710, 10240, 11900 and 20760 were investigated.

# 1 Particle motion equation

The particle motion in turbulent flow is acted by the relative drags, pressure gradient force, added mass force, Basset force, gravity, buoyancy force, Magus and Saffma forces and so on. In the main stream area, Magus and Saffman forces can be neglected. The following Lagrangian equation for particle motion in turbulent flow can be obtained according to the Newton's Second Law.

$$\frac{du_{pi}}{dt} = \frac{\rho_f}{\rho_p}\frac{du_{fi}}{dt} + \frac{3}{4}C_{Di}\frac{\rho_f}{\rho_p}\frac{|u_{fi} - u_{pi}|}{d_p}(u_{fi} - u_{pi}) + k_{mi}\frac{\rho_f}{\rho_p}\left|\frac{du_{fi}}{dt} - \frac{du_{pi}}{dt}\right| + \frac{3}{2}k_{Bi}\frac{\rho_f}{\rho_p}\sqrt{\frac{\upsilon}{\pi}}\frac{1}{d_p}\int_{-\infty}^{t}\frac{\frac{du_{fi}}{d\tau} - \frac{du_{pi}}{d\tau}}{\sqrt{t-\tau}}d\tau + \left(\frac{\rho_p - \rho_f}{\rho_p}\right)g_i. \tag{1}$$

Where, $u_{pi}$, $u_{fi}$ are the longitudinal ($i = 1$) and vertical ($i = 2$) velocities of particle and fluid respectively; $d_p$ the diameter of particle; $\rho_p$, $\rho_f$ the densities of particle and fluid; $v$ the kinetic viscosity of fluid; $C_{Di}$, $k_{mi}$, $k_{Bi}$ the coefficients of the relative drag, added mass force and Basset history force; $g_i$ the acceleration of gravity. The terms on the right of this equation are pressure gradient force, relative drag, added mass force, Basset force, gravity and buoyancy force.

Eq.(1) can be rewritten as follows:

$$\frac{du_{pi}}{dt} = \frac{3C_{Di}\rho_f\upsilon Rep_i}{4d_p^2(\rho_p + k_{mi}\rho_f)}(u_{fi} - u_{pi}) + \frac{\rho_f(1 - k_{mi})}{\rho_p + k_{mi}\rho_f}\frac{du_{fi}}{dt} + \frac{\rho_p - \rho_f}{\rho_p + k_{mi}\rho_f}g_i + \frac{3k_{Bi}\rho_f\sqrt{\nu/\pi}}{2d_p(\rho_P + k_{mi}\rho_f)}\int_{-\infty}^{t}\frac{(du_{fi}/d\tau - du_{pi}/d\tau)}{\sqrt{t-\tau}}d\tau. \tag{2}$$

In which

$$C_{Di} = \begin{cases} 24/Rep_i & Rep_i \leqslant 1 \\ (24/Rep_i)(1 - 0.15Rep_i^{0.68}) & 1 < Rep_i < 1000 \\ 0.44 & Rep_i \geqslant 1000 \end{cases}$$

$$k_{mi} = 1.05 - 0.066/(AC_i^2 + 0.12);$$

$$k_{Bi} = 2.88 + 3.12/(AC_i + 1.0)^3;$$

$$Rep_i = d_p|u_{fi} \quad u_{pi}|/\nu,\ AC = (u_{fi} - u_{pi})^2/d_p\left|\frac{du_{fi}}{dt} - \frac{du_{pi}}{dt}\right|.$$

Eq.(2) can be written as follows:

$$\frac{du_{pi}}{dt} = A_i(u_{fi} - u_{pi}) + B_i\frac{du_{fi}}{dt} + C_i\int_{-\infty}^{t}\frac{(du_{fi}/d\tau - du_{pi}/d\tau)}{\sqrt{t-\tau}}d\tau + D_ig_i. \tag{3}$$

In which

$$A_i = \frac{3C_{Di}\rho_f\upsilon Rep_i}{4d_p^2(\rho_p + k_{mi}\rho_f)},\quad B_i = \frac{\rho_f(1 + k_{mi})}{\rho_p + k_{mi}\rho_f},$$

$$C_i = \frac{3k_{Bi}\rho_f\sqrt{\nu/\pi}}{2d_p(\rho_P + k_{mi}\rho_f)}, \quad D_i = \frac{\rho_p - \rho_f}{\rho_p + k_{mi}\rho_f}.$$

Considering two dimensional motions, the longitudinal and vertical motions of particle, the following equation can be obtained from Eq. (3) by dividing the fluid velocity into the time-mean velocity and fluctuation velocity($u_{f1} = \bar{u}_{f1} + u'_{f1}$) and defining the velocity as zero at the integral domain of $t = -\infty \sim 0$.

$$\left.\begin{aligned} \frac{du_{p1}}{dt} &= A_1(\bar{u}_{f1} + u'_{f1} - u_{p1}) + B_1\frac{du_{f1}}{dt} + C_1\int_0^t \frac{du_{f1}/d\tau - du_{p1}/d\tau}{\sqrt{t-\tau}}d\tau \\ \frac{du_{p2}}{dt} &= A_2(u_{f2} + u'_{f2} - u_{p2}) + B_2\frac{du_{f2}}{dt} - C_2\int_0^t \frac{du_{f2}/d\tau - du_{p2}/d\tau}{\sqrt{t-\tau}}d\tau + D_2 g\ . \end{aligned}\right\} \quad (4)$$

## 2 Calculation of flow field

### 2.1 Time-mean velocity of fluid

The vertical time-mean velocity can be taken as zero ($\bar{u}_{f2} = 0$). The longitudinal time-men velocity $\bar{u}_{f1}$ is calculated based on the following formula deduced by Dou (Dou, 1987) from Stochastic theory for the homogeneous shear turbulent flows in two dimensional open channel. Fig. 1 shows that the calculated results are good accordance with the experimental results.

$$\frac{\bar{u}_{f1}}{u_*} = a\ln\left(1 + \frac{u_* y}{2av}\right) + b\left(\frac{u_* y}{2av}\Big/\left(1 + \frac{u_* y}{2av}\right)\right)^2 + a\left(\frac{u_* y}{2av}\Big/\left(1 + \frac{u_* y}{2av}\right)\right). \quad (5)$$

Where, $u_*$ ($= \sqrt{gRj}$) is the frictional velocity, $R$ the hydraulic radius, $j$ the base slope; $y$ the distance from the bottom of the channel; $a$ and $b$ are constants; determined by the experimental data.

### 2.2 Fluctuation velocity

The fluctuation velocities $u'_{f1}$, $u'_{f2}$ are modelled by the random Fourier series.

$$\begin{aligned} u'_{f1} &= \sum_1^n R_1 A_{1i}\cos(\omega_i t + R_2 2\pi), \\ u'_{f2} &= \sum_1^n R_3 A_{2i}\cos(\omega_i t + R_4 2\pi). \end{aligned} \quad (6)$$

In which, $R_1$ and $R_3$ are the random number obeying the normal distribution; $R_2$ and $R_4$ are the random number obeying the (0, 1) uniform distribution; $\omega_i$ is the angle frequency; $A_{1i}$ and $A_{2i}$ are the turbulent fluctuation amplitudes of the eddy with the angle frequency $\omega_i$, determined according to the turbulent fluctuation spectra.

$A_{1i}$ and $A_{2i}$ are calculated by the following expressions:

$$\begin{aligned} A_{1i}^2 &= k_{1i}\overline{u_{f1}'^2} \\ A_{2i}^2 &= k_{2i}\overline{u_{f2}'^2}, \end{aligned} \quad (7)$$

where $k_{1i}$ and $k_{2i}$ are the longitudinal and vertical turbulent fluctuation energy's percents of the eddy with the angle frequency $\omega_i$ in the total turbulent fluctuation energies respectively, calculated by the following expressions:

$$\begin{aligned} k_{1i} &= \frac{E_1(\omega_i)\Delta\omega_i}{\sum E_1(\omega_i)\Delta\omega_i}, \\ k_{2i} &= \frac{E_2(\omega_i)\Delta\omega_i}{\sum E_2(\omega_i)\Delta\omega_i}. \end{aligned} \quad (8)$$

In the homogeneous shear turbulent flows, the longitudinal and vertical fluctuation spectra $E_1(\omega)$ and $E_2(\omega)$ can be calculated using the following expressions:

$$\frac{E_1(\omega)}{\overline{u_{f1}'^2}} = \sqrt{\frac{2}{3\pi}}\frac{1}{\bar{\omega}}\left[1 - \frac{b_*}{3}\left(1 - \frac{\omega^2}{3\bar{\omega}^2}\right)\right]\exp\left(-\frac{\omega^2}{6\bar{\omega}^2}\right),$$

$$\frac{E_2(\omega)}{\overline{u_{f2}'^2}} = \sqrt{\frac{2}{3\pi}}\frac{1}{\bar{\omega}}\left[1 - \frac{c_*}{3}\left(1 - \frac{\omega^2}{3\bar{\omega}^2}\right)\right]\exp\left(-\frac{\omega^2}{6\bar{\omega}^2}\right). \tag{9}$$

In which

$$\bar{\omega}^2 = \frac{1}{3}(2\omega_*^2 + \omega_*^{1/2}\omega^{3/2}), \quad \omega_* - \frac{\bar{u}_{f1}}{H}, \tag{10}$$

$$b_* = \frac{B_2}{B_1} = \frac{B_2}{\bar{u}_{f1}'^2}, \quad C_* = \frac{C_2}{C_1} = \frac{C_2}{\bar{u}_{f2}'^2}, \tag{11}$$

$$\begin{aligned}\frac{B_2}{u_*^2} = {} & \frac{1}{4}\left(\frac{u_* y/v}{d + u_* y/v}\right) - \frac{1}{2a}\sqrt{1 - \frac{y}{H}}\left[\sqrt{1 - \frac{y}{H}} + 2\sqrt{\frac{u_* y/v}{d + u_* y/v}}\left(1 + \frac{d/8}{u_* y/v}\right)\right]\left(\frac{y}{u_*}\frac{d\bar{u}_{f1}}{dy}\right) \\ & + \frac{1}{a^2}\left(1 - \frac{y}{H}\right)\left[1 + \sqrt{\left(1 - \frac{y}{H}\right)\left(1 + \frac{d}{u_* y/v}\right)} + \frac{d/4}{u_* y/v}\right]\left(\frac{y}{u_*}\frac{d\bar{u}_{f1}}{dy}\right)^2,\end{aligned} \tag{12}$$

$$\frac{C_2}{u_*^2} = \frac{1}{4}\left(\frac{u_* y/v}{d + u_* y/v}\right) - \frac{d/8a}{u_* y/v}\sqrt{\left(1 - \frac{y}{H}\right)\left(\frac{u_* y/v}{d + u_* y/v}\right)}\left(\frac{y}{u_*}\frac{d\bar{u}_{f1}}{dy}\right), \tag{13}$$

$$\frac{y}{u_*}\frac{d\bar{u}_{f1}}{dy} = \frac{u_* y/v}{1 + u_* y/2av} + \frac{d/4a^2(u_* y/v)^2 - 1/8a^2(u_* y/v)^3}{(1 + u_* y/2av)^3}. \tag{14}$$

Where $d = 2b - a$ is a constant, $H$ the water depth.

$$\frac{\overline{u_{f1}'^2}}{u_*^2} = \frac{u_* y/2v}{u_* y/v + d} + \frac{9}{4a^2}\left(1 - \frac{y}{H}\right)\left[\frac{u_* y/v}{1 + u_* y/2av} + \frac{d/4a^2(u_* y/v)^2 - 1/8a^2(u_* y/v)^3}{(1 + u_* y/2av)^3}\right]^2, \tag{15}$$

$$\frac{\overline{u_{f2}'^2}}{u_*^2} = \frac{u_* y/2v}{u_* y/v + d} + \frac{1}{4a^2}\left(1 - \frac{y}{H}\right)\left[\frac{u_* y/v}{1 + u_* y/2av} + \frac{d/4a^2(u_* y/v)^2 - 1/8a^2(u_* y/v)^3}{(1 + u_* y/2av)^3}\right]^2. \tag{16}$$

## 3 Solution to particle motion equation

Eq.(4) can be solved using the fourth order Runge-Kutta methods. The particle position was determined by the following expressions

$$\begin{aligned} x &= x_0 + u_{p1}\Delta t, \\ y &= y_0 + u_{p2}\Delta t. \end{aligned} \tag{17}$$

In which, the subscript "0" indicates the value at the last time step, $\Delta t$ the time step size, determined by the following expression:

$$\Delta t \leqslant \min(\tau_l, \tau_R), \tag{18}$$

where, $\tau_l$ is the existing time of random eddy, $\tau_R$ are time of the particle passing through random eddy, which are determined referred to literature(Zhang, 1997).

After the Basset history force is calculated, the particle motion velocity can be obtained by solving Eq.(4) using the fourth order Runge-Kutta method, and then the trajectory of particle can be calculated using Eq.(17).

## 4 Analyses of results

This experiment is conducted in a rectangular channel with the length of 15m, width of 15cm and the adjustable slope. The flow conditions with different flow Reynolds number are obtained by changing the discharge. The selected suspended particles are required not to dissolve and distort in water, that is, the density and volume are constant. In this experiment, the spherical cation exchange resins with a density of 1.44 g/cm$^3$ and diameters ranging from 0.50—0.60 mm, 0.60—0.70 mm and 0.80—0.90 mm are selected as the experimental solid particles.

The time-mean velocity and the r.m.s. of fluid are measured using the "two dimensional optic fiber Doppler velometer" produced by DANTEC Company, Denmark. In a given measured section, the velocities of fluid without particles are first measured by the laser Doppler velometer, then the suspended particles were fed continuously by a feeding funnel in the upper reach of the given measured section, and the moving processes of suspended particles in measured section are photographed by a camera. The photographed images are input into the computer and processed using the digital image processing technology, then the moving velocity and trajectory of suspended particle can be obtained.

The flows with the flow Reynolds number of 4710, 10240, 11900 and 20760 are selected as the experimental flow conditions.

### 4.1 Fluid velocity

For every flow condition, two control sections are arranged at the upper and down reaches of the measured section respectively, and five vertical measured lines with eight measured points are laid out in every control section.

From Fig.1, it can be seen that the time-mean velocity of the upper control section is approximately equal to that of the down control section at the same water depth, so the flow can be approximated as uniform flow, and the time-mean velocity can be calculated using Eq. (5). The constants a and b are fitted as $a = 0.85$, $b = 13.5$ according to the experimental data. Fig.1 has followed that Eq. (5) can describe well the time-mean velocity distribution of the selected experimental flow conditions.

The ratio of r. m. s. to the frictional velocity $\sqrt{u_f'^2}/u_*$ is plotted versus the dimensionless water depth y/H in Fig.2. It can be seen that Eq. (15) can also well describe the fluctuation velocities of the experimental flow conditions.

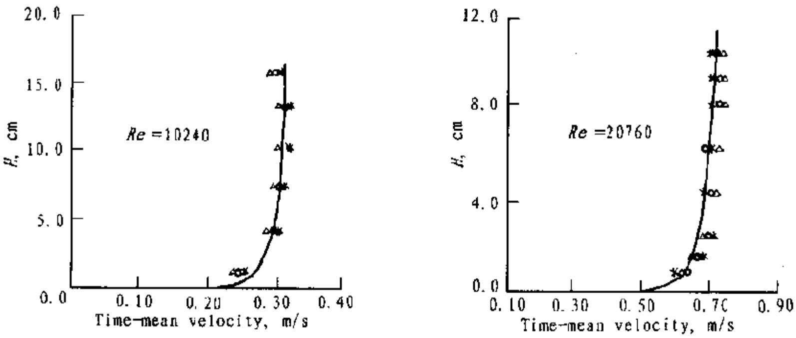


Fig. 1 The velocity distribution of middle vertical measured line at different flow conditions
○ △ ☆ measured values; —— calculated values by Eq. (5)

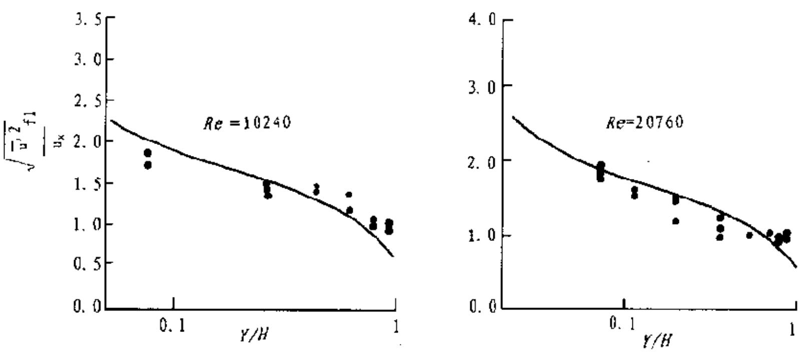


Fig. 2 The fluctuation intensity of fluid varies with water depth
●measured values; ——calculated values by Eq. (15)

### 4.2 Trajectory of particle

The measured and calculated trajectories of different particles in different flow conditions are plotted in Fig.3. It can be seen that the FSRTM developed in this paper can well describe the particle motion in turbulent flows, in which the motion essence of eddies with different periods, amplitudes and phase angles are considered, so the effect of the turbulent fluctuation on particle motion is taken into account fully. In the flow with flow Reynolds number of 4710, because the fluctuation of fluid is weak, the trajectory of particle is very smooth, and the settling motions of all particles due to their higher settling velocities of gravity are obvious. For the particle with diameter ranging from 0.80 to 0.90 mm, its settling velocity due to gravity is higher than the fluctuation velocity of fluid, even in the flow with Reynolds number of 20760, its settling motion is very obvious. In the measured section having length of 20 cm, the particle with diameter ranging from 0.80 to 0.90 mm has a settling distance of 4 cm, the particle with $d_p = 0.50—0.60$ mm also has a settling distance of 1 cm. Owing to the higher turbulent intensity of fluid, even the smaller particle is affected obviously and has a sinuous moving trajectory. In the same flow condition, the ability of particle to respond the turbulence of fluid increases with the decrease of particle's diameter, and therefore the irregularity of the moving trajectory of particle increases. For the same particle, the irregularity of particle motion increases with the increase of the turbulence of fluid. But for the particle with the higher density, when its settling velocity due to gravity is higher than the fluctuation velocity of fluid, it has an obvious settling motion, that is, in the main stream direction, the time-mean velocity of fluid controls the particle motion.

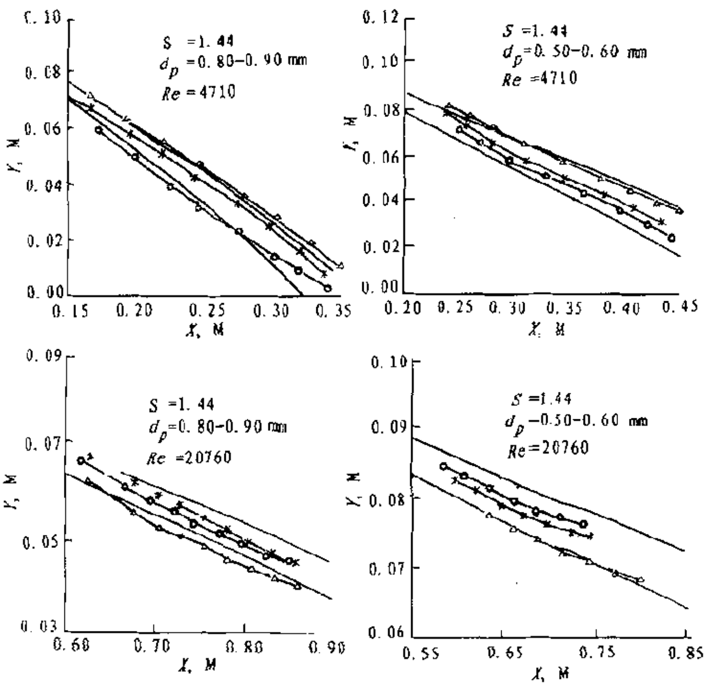


Fig.3 The moving trajectories of suspended particles in different flow conditions
• ○ ☆ ▵ measured trajectory; —— calculated trajectory by FSRTM

### 4.3 Moving velocity of particle

The time-mean velocity of fluid and the measured and calculated longitudinal moving velocities of particle are plotted in Fig.4. It can be seen that the moving velocity of particle calculated by

FSRTM is good accordance with the measured value. In the flow with weak turbulence of fluid ($Re = 4710$), the difference between the moving velocity of particle and the time-mean velocity of fluid is small. But with the increase of the turbulence of fluid, the relative velocity of particle and fluid increases, and the two-phase velocity difference depends largely on the time-mean velocity of fluid. When the turbulent intensity is of fluid high, the moving velocity of particle has an intensive fluctuation. In the same flow condition, the velocity difference of particle and fluid increases with the increase of particle's diameter. It can also be seen from Fig. 4 that the moving velocity of particle tends immediately to equal that of fluid after particle being fed into water, which follows that the effect of particle's initial condition on its following motion to fluid is short and can be neglected. The flowing condition of fluid, particle inertia and the "crossing trajectory effect" due to particle's gravity control the particle motion in turbulent flows. The particle inertia tends to maintain the following motion of particle to fluid, but the crossing trajectory effect tends to reduce the following motion of particle.

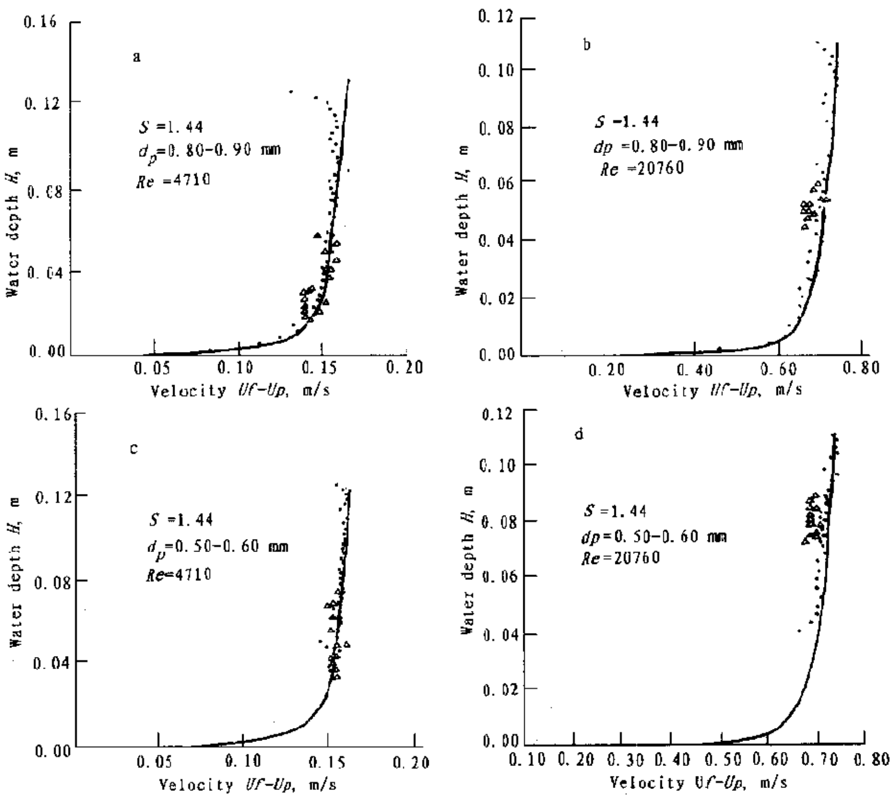


Fig. 4 The longitudinal moving velocities of suspended particles in different flows
——Time-mean velocity of fluid
△ measured particle velocity; ● calculated particle velocity by FSRTM

# 6 Conclusions

The fluctuation spectrum random trajectory model (FSRTM) for the particle motion in environmental fluid is developed using Lagrangian method, in which the essence of turbulent eddy is considered fully, that is, the turbulence is composed randomly of the eddies which different frequencies, amplitudes and phase angles. This model is used to calculate the moving trajectories and velocities of different particles in different flow conditions. The numerical calculation result are well accordance with the experimental results, which follows that the FSRTM can describe well the particle motion in turbulent flows.

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